VI. *Addition to the Memoir on* Tschirnhausen's *Transformation.* *By* Arthur Cayley, *F.R.S.*

In the memoir "On Tschirnhausen's Transformation," Philosophical Transactions, vol. clii. (1862) pp. 561–568, I considered the case of a quartic equation: viz. it was shown that the equation

$$(a, b, c, d, e\!\!\;\rangle\!\!\;x, 1)^4 = 0$$

is, by the substitution

$$y = (ax + b)\mathrm{B} + (ax^2 + 4bx + 3c)\mathrm{C} + (ax^3 + 4bx^2 + 6cx + 3d)\mathrm{D},$$

transformed into

$$(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E}\!\!\;\rangle\!\!\;y, 1)^4 = 0,$$

where $(\mathbb{C}, \mathbb{D}, \mathbb{E})$ have certain given values. It was further remarked that $(\mathbb{C}, \mathbb{D}, \mathbb{E})$ were expressible in terms of U', H', Φ', invariants of the two forms $(a, b, c, d, e\!\!\;\rangle\!\!\;\mathrm{X}, \mathrm{Y})^4$, $(\mathrm{B}, \mathrm{C}, \mathrm{D}\!\!\;\rangle\!\!\;\mathrm{Y}, -\mathrm{X})^2$, of I, J, the invariants of the first, and of Θ', $=\mathrm{BD} - \mathrm{C}^2$, the invariant of the second of these two forms,—viz. that we have

$$\mathbb{C} = 6\mathrm{H}' - 2\mathrm{I}\Theta',$$

$$\mathbb{D} = 4\Phi',$$

$$\mathbb{E} = \mathrm{I}\mathrm{U}'^3 - 3\mathrm{H}'^2 + \mathrm{I}^2\Theta'^2 + 12\mathrm{J}'\Theta'\mathrm{U}' + 2\mathrm{I}'\Theta'\mathrm{H}'.$$

And by means of these I obtained an expression for the quadrinvariant of the form

$$(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E}\!\!\;\rangle\!\!\;y, 1)^4;$$

viz. this was found to be

$$= \mathrm{I}\mathrm{U}'^2 + \tfrac{4}{3}\mathrm{I}^2\Theta'^2 + 12\mathrm{J}\Theta'\mathrm{U}'.$$

But I did not obtain an expression for the cubinvariant of the same function: such expression, it was remarked, would contain the square of the invariant Φ'; it was probable that there existed an identical equation,

$$\mathrm{J}\mathrm{U}'^3 - \mathrm{I}\mathrm{U}'^2\mathrm{H}' + 4\mathrm{H}'^3 + \mathrm{M}\Theta' = -\Phi'^2,$$

which would serve to express Φ'^2 in terms of the other invariants; but, assuming that such an equation existed, the form of the factor M remained to be ascertained; and until this was done, the expression for the cubinvariant could not be obtained in its most simple form. I have recently verified the existence of the identical equation just referred to, and have obtained the expression for the factor M; and with the assistance of this identical equation I have obtained the expression for the cubinvariant of the form

$$(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E}\!\!\;\rangle\!\!\;y, 1)^4.$$

The expression for the quadrinvariant was, as already mentioned, given in the former memoir: I find that the two invariants are in fact the invariants of a certain linear function of U, H; viz. the linear function is $= U'U + \frac{2}{3}\Theta'H$; so that, denoting by I*, J*, the quadrinvariant and the cubinvariant respectively of the form

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \,\backslash\!\!\backslash y, 1)^4,$$

we have

$$I* = \tilde{I}(U'U + 4\Theta'H),$$

$$J* = \tilde{J}(U'U + 4\Theta'H),$$

where $\tilde{I}$, $\tilde{J}$ signify the functional operations of forming the two invariants respectively. The function $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \,\backslash\!\!\backslash y, 1)^4$, obtained by the application of TSCHIRNHAUSEN's transformation to the equation

$$(a, b, c, d, e \,\backslash\!\!\backslash x, 1)^4 = 0,$$

has thus the *same invariants* with the function

$$U'U + 4\Theta'H = U'(a, b, c, d, e \,\backslash\!\!\backslash x, 1)^4 + 4\Theta'(ac - b^2, ad - bc, ae + 2bd - 3c^2, be - cd, ce - d^2 \,\backslash\!\!\backslash x, 1)^4,$$

and it is consequently a linear transformation of the last-mentioned function; so that the application of TSCHIRNHAUSEN's transformation to the equation $U = 0$ gives an equation linearly transformable into, and thus virtually equivalent to, the equation

$$U'U + 4\Theta'H = 0,$$

which is an equation involving the single parameter $\dfrac{4\Theta'}{U'}$: this appears to me a result of considerable interest. It is to be remarked that TSCHIRNHAUSEN's transformation, wherein y is put equal to a rational and integral function of the order $n-1$ (if n be the order of the equation in x), is not really less general than the transformation wherein y is put equal to any rational function $\dfrac{V}{W}$ whatever of x; such rational function may, in fact, by means of the given equation in x, be reduced to a rational and integral function of the order $n-1$; hence in the present case, taking V, W to be respectively of the order $n-1$, $= 3$, it follows that the equation in y obtained by the elimination of x from the equations

$$(a, b, c, d, e \,\backslash\!\!\backslash x, 1)^4 = 0,$$

$$y = \frac{(\alpha, \beta, \gamma, \delta \,\backslash\!\!\backslash x, 1)^3}{(\alpha', \beta', \gamma', \delta' \,\backslash\!\!\backslash x, 1)^3}$$

is a mere linear transformation of the equation $AU + BH = 0$, where A, B are functions (not as yet calculated) of $(a, b, c, d, e, \alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta')$.

Article Nos. 1, 2, 3.—*Investigation of the identical equation*

$$JU'^3 - IU'^2H' + 4H'^3 + M\Theta' = -\Phi'^2.$$

1. It is only necessary to show that we have such an equation, M being an invariant,

in the particular case $a=e=1$, $b=d=0$, $c=\theta$, that is for the quartic function $(1, 0, \theta, 0, 1 \mathbb{X} x, 1)^4$; for, this being so, the equation will be true in general. Writing the equation in the form

$$-M\Theta'=U'^2(JU'-IH')+4H'^3+\Phi'^2,$$

and observing that we have

$$U'=(B^2+D^2)+2\theta BD+4\theta C^2,$$

$$H'=\theta(B^2+D^2)+(1+\theta^2)BD-4\theta^2C^2,$$

$$\Theta'=BD-C^2,$$

$$\Phi'=(1-9\theta^2)C(B^2-D^2),$$

$$I=1+3\theta^2,$$

$$J=\theta-\theta^3,$$

and thence

$$JU'-IH'=-4\theta^3(B^2+D^2)+(-1-2\theta^2-5\theta^4)BD+(8\theta^2+8\theta^4)C^2,$$

the equation becomes

$$-(BD-C^2)M=$$

$$\{-4\theta^3(B^2+D^2)+(-1-2\theta^2-5\theta^4)BD+(8\theta^2+8\theta^4)C^2\}$$
$$\times\{B^2+D^2+2\theta BD+4\theta C^2\}^2$$
$$+4\{\theta(B^2+D^2)+(1+\theta^2)BD-4\theta^2C^2\}^3$$
$$+(1-9\theta^2)^2 C^2\{(B^2+D^2)^2-4B^2D^2\}.$$

2. It is found by developing that the right-hand side is in fact divisible by $BD-C^2$, and that the quotient is

$$= \quad (-1+10\theta^2-9\theta^4)(B^2+D^2)^2$$
$$+(8\theta+16\theta^3-24\theta^5)(B^2+D^2)BD$$
$$+(4+8\theta^2+4\theta^4-16\theta^6)B^2D^2$$
$$+(-64\theta^3-192\theta^5)(B^2+D^2)C^2$$
$$+(16\theta^2-416\theta^4-112\theta^6)BDC^2$$
$$+(-128\theta^4+128\theta^6)C^4.$$

3. This is found to be

$$= -I^2U'^2+12JU'H'+4IH'^2$$
$$- 8IJU'\Theta'$$
$$-16J^2\Theta'^2,$$

which is consequently the value of $-M$. We have therefore

$$-\Phi'^2= \quad JU'^3-IU'^2H'+4H'^3$$
$$+(I^2U'^2-12JU'H'-4IH'^2)\Theta'$$
$$+ 8IJU'\Theta'^2$$
$$+16J^2\Theta'^3,$$

which is the required identical equation.

Article No. 4.—*Calculation of the Cubinvariant.*

4. We have

$$J^* = \tfrac{1}{6}\mathbb{C} \cdot \mathbb{E} - (\tfrac{1}{6}\mathbb{C})^3 - (\tfrac{1}{4}\mathbb{D})^2$$
$$= (H - \tfrac{1}{3}I\Theta')\{IU'^2 - 3H'^2 + (12JU' + 2IH')\Theta' + I^2\Theta'^2\}$$
$$- (H - \tfrac{1}{3}I\Theta')^3$$
$$- \Phi'^2,$$

whence, substituting for $-\Phi'^2$ its value and reducing, we find

$$J^* = JU'^3 + \Theta' \cdot \tfrac{2}{3}I^2U'^2 + \Theta'^2(4IJU') + \Theta'^3(16J^2 - \tfrac{8}{27}I^3).$$

Article No. 5.—*Final expressions of the two Invariants.*

The value of I^* has been already mentioned to be $I^* = IU'^2 + \Theta'12JU' + \Theta'^2 \cdot \tfrac{4}{3}I^2$, and it hence appears that the values of the two invariants may be written

$$I^* = (I, 18J, 3I^2 \, \check{\jmath} \, U', \tfrac{2}{3}\Theta')^2,$$
$$J^* = (J, I^2, 9IJ, -I^3 + 54J^2 \, \check{\jmath} \, U', \tfrac{2}{3}\Theta')^3.$$

But we have (see Table No. 72 in my "Seventh Memoir on Quantics" †)

$$\tilde{I}(\alpha U + 6\beta H) = (I, 18J, 3I \, \check{\jmath} \, \alpha, \beta)^2$$
$$\tilde{J}(\alpha U + 6\beta H) = (J, I^2, 9IJ, -I^3 + 54J^2 \, \check{\jmath} \, \alpha, \beta)^3;$$

so that, writing $\alpha = U'$, $\beta = \tfrac{2}{3}\Theta'$, we have

$$I^* = \tilde{I}(U'U + 4\Theta'H),$$
$$J^* = \tilde{J}(U'U + 4\Theta'H);$$

or the function $(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E} \, \check{\jmath} \, y, 1)^4$ obtained from TSCHIRNHAUSEN'S transformation of the equation $U = 0$ has the same invariants with the function $U'U + 4\Theta'H$; or, what is the same thing, the equation $(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E} \, \check{\jmath} \, y, 1^4) = 0$ is a mere linear transformation of the equation $U'U + 4\Theta H = 0$; which is the above-mentioned theorem.

† Philosophical Transactions, vol. cli. (1861), pp. 277–292.